Point Mass on a Circular Ramp

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1 Introduction

We consider here the simple mechanics problem of a point mass sliding down a circular ramp. Although this can be solved easily using Newtonian mechanics, we will use the Lagrangian formalism for practice. The effect of friction will be considered.

2 Formal Analysis

We assume the mass starts at rest at the top of a quarter-circle ramp, as shown in Figure 1. The angular coordinate θ is defined as shown. In particular, $\theta = 0$ is at the top of the ramp and $\theta = \frac{\pi}{2}$ is at the bottom. The cartesian x and y coordinates can be expressed in terms of θ as follows.

$$\begin{aligned} x &= r\cos\theta & (1) \\ y &= r - r\sin\theta = r\,(1-\sin\theta) & (2) \end{aligned}$$

In these coordinates, we can easily write the kinetic and potential energies as well as the Lagrangian.

$$\begin{aligned} T &= \frac{1}{2}mr^2\dot{\theta}^2 & (3) \\ V &= -mgr\sin\theta & (4) \\ L &= T - V = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\sin\theta & (5) \end{aligned}$$

We would also like to consider friction. Because this is a dissipative force, we cannot include it in the potential. We will assume the friction force is given by a coefficient, γ, times the force normal to the surface, in the direction opposite the mass's velocity. The magnitude of the normal force is

$$F_\perp = mg\sin\theta \tag{6}$$

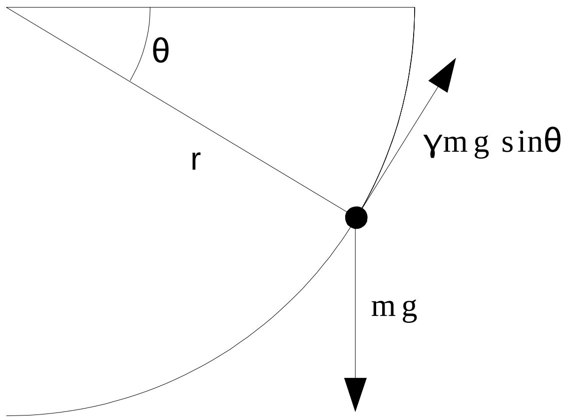

Figure 1: Physical configuration of the problem.

so the actual friction force is

$$\begin{aligned}\vec{F}_f &= \gamma F_\perp \sin\theta \hat{x} + \gamma F_\perp \cos\theta \hat{y} \\ &= \gamma mg \sin^2\theta \hat{x} + \gamma mg \sin\theta \cos\theta \hat{y} \qquad (7)\end{aligned}$$

Thus, the generalized friction force is

$$\begin{aligned}Q_f &= F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta} \\ &= \left(\gamma mg \sin^2\theta\right)(-r\sin\theta) + (\gamma mg \sin\theta\cos\theta)(-r\cos\theta) \\ &= -\gamma mgr \sin\theta \left(\cos^2\theta + \sin^2\theta\right) \\ &= -\gamma mgr \sin\theta \qquad (8)\end{aligned}$$

where we have assumed the particle will always be moving in the $+\theta$ direction.

We can now write Lagrange's equation of motion for our system, including the nonconservative friction force.

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= Q_f \\ \frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgr\cos\theta &= -\gamma mgr \sin\theta \\ mr^2\ddot{\theta} &= mgr\left(\cos\theta - \gamma\sin\theta\right) \\ \ddot{\theta} &= \frac{g}{r}\left(\cos\theta - \gamma\sin\theta\right) \qquad (9)\end{aligned}$$

This final expression is the differential equation that describes the motion of the mass. In the absence of friction, this can be solved exactly and describes the motion of a pendulum with large displacements.

For our purposes, we will be solving this equation numerically. To simplify this, we will rescale the time coordinate to eliminate the $\frac{g}{r}$ on the right-hand side of our equation. If we change our time coordinate to

$$t' = t\sqrt{\frac{g}{r}} \tag{10}$$

then we have

$$\begin{aligned} \ddot{\theta} &= \frac{d}{dt}\left(\frac{d\theta}{dt}\right) \\ &= \frac{d}{dt'}\frac{dt'}{dt}\left(\frac{d\theta}{dt'}\frac{dt'}{dt}\right) \\ &= \frac{g}{r}\frac{d^2\theta}{dt'^2}. \end{aligned} \tag{11}$$

If we now let dots indicate differentiation with respect to t', our differential equation becomes

$$\ddot{\theta} = \cos\theta - \gamma\sin\theta \tag{12}$$

which is unitless and convenient for numerical treatment.

3 Numerical Solution

To study the behavior of the sliding mass with different degrees of friction, we numerically solved (12) with values of γ between 0.0 and 1.0 in steps of 0.1. The resulting curves are plotted in Figure 2 starting at $\theta = 0$ and ending at $\theta = \frac{\pi}{2}$, when the mass reaches the end of the ramp.

For $\gamma = 0$, the left-most curve, we see that the mass initially accelerates before reaching a uniform velocity near the end of the ramp. This is intuitively sensible behavior — at the bottom of the ramp, gravity is perpendicular to the surface so no further acceleration occurs.

When γ is increased from 0, the initial portion of the curve is unchanged. This is to be expected because initially, the normal force from the surface is 0 so friction has no effect. As the mass reaches the bottom of the ramp, the normal force increases so there is a larger deviation from the frictionless solution.

When $\gamma = 1$, the mass only asymptotically reaches $\theta = \frac{\pi}{2}$. This is evident from the curve, but we can show explicitly that this should be so. The total kinetic energy of the mass when it reaches angle θ_0 is given by

$$\begin{aligned} T &= V(0) - V(\theta_0) + \int_0^{\theta_0} Q_f(\theta)d\theta \\ &= mgr\left(1 - \cos\theta_0\right) - \int_0^{\theta_0} \gamma mgr\sin\theta. \end{aligned} \tag{13}$$

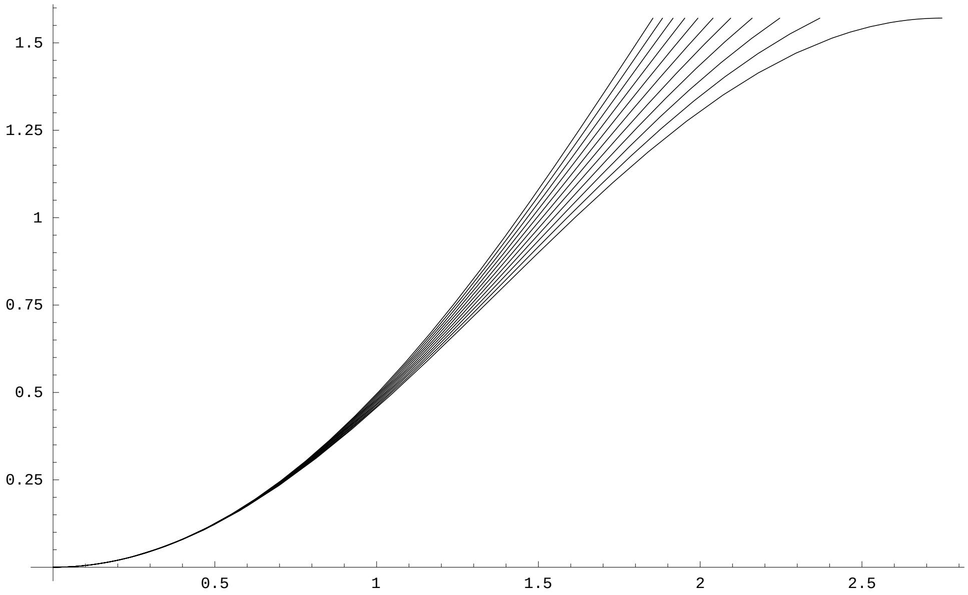

Figure 2: Solutions with various values for γ. The x-axis is normalized time ($t\sqrt{\frac{g}{r}}$) and the y-axis is θ. The left-most curve is $\gamma = 0$ and γ increases to the right.

Setting $\gamma = 1$ and $\theta_0 = \frac{\pi}{2}$ we get

$$\begin{aligned} T &= mgr\left(1 - \int_0^{\pi/2} \sin\theta d\theta\right) \\ &= mgr\left(1 + \cos\frac{\pi}{2} - \cos 0\right) \\ &= 0. \end{aligned} \tag{14}$$

When $\gamma = 1$, the mass dissipates all of its potential energy against the friction force on its way down and comes to a rest precisely at the bottom of the circular ramp.

4 Conclusions

We have solved the simple mechanics problem successfully and found that its solutions make physical sense. The Lagrangian formalism was probably overkill in this case — we could have skipped the Lagrangian entirely by recognizing this as a problem of angular momentum and torques. In fact, the Lagrange equations we wrote down corresponded exactly to the angular momentum analysis.

We only analyzed solutions for $\gamma <= 1.0$. If γ is increased beyond 1.0, the mass will dissipate its energy before it reaches the bottom of the ramp. As a result, it will not actually reach $\theta = \frac{\pi}{2}$. Instead, the mass will slide to a point slightly below the highest balance point where friction's resistance can just balance gravity's pull. Unfortunately, our equations cannot simply be integrated to demonstrate this. Equation (8) for Q_f presumed that our mass was always moving in the $+\theta$ direction. Once the mass stops, this force should vanish — friction cannot accelerate the particle up the ramp, yet a numerical solution to our equation of motion will exhibit just this behavior!

To avoid this catastrophic breakdown of our mathematics, we need to write a more accurate analytic expression for Q_f. It is not immediately obvious how to do this. The friction we speak of acts to reduce velocity and to prevent acceleration from rest but must be exactly balanced to avoid accidentally accelerating the mass from restin either direction. It is thus a function not only of the coordinates of the mass, but also the other forces acting on the mass.

Regardless, our energy arguments clearly explain the behavior of the mass for $\gamma > 1.0$. The angle to which the mass falls before stopping can be found analytically from (13) or numerically by solving our equation of motion and identifying the largest θ it reaches before starting back up the ramp as the last physically meaningful point on the trajectory.